

Signs and Polarized/Magnetic versions of the Casimir Forces

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 (TP-95-004, quant-ph/9504019, February 1, 2008)

We consider versions of the Casimir effect where the force can be controlled by changing the angle between two Casimir “plates” or the temperature of two nearby rings. We also present simple arguments for the sign of Casimir forces.

The attractive Casimir force: [1]

$$F_{cas} = -\frac{\pi^2}{240} \frac{\hbar c}{a^4} \frac{dynes}{cm} \quad (1)$$

acts between two parallel plates at a distance a apart in vacuum. Its independence of atomic parameters and the QED coupling, α_E , reflects the perfect conductor idealization. All details are subsumed into the boundary conditions

$$E_x = 0, E_y = 0, \text{ at } z = 0, \text{ or } z = a, \quad (2)$$

imposed on the transverse vacuum fields. This in turn quantizes $k_z = \frac{n\pi}{a}$, the z component of wave number for modes in the region between the plates. The problem then reduces to evaluation of the change in vacuum energy of all the transverse modes inside this region:

$$\frac{1}{2}\hbar c \left\{ \int dk_x dk_y \left(\sum_n \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{a}\right)^2} - \int dk_z \sqrt{k_x^2 + k_y^2 + k_z^2} \right) \right\}. \quad (3)$$

A careful regularization of this formally divergent expression yields [2]

$$E_{cas}(a) = -\frac{\pi^2 \hbar c}{720 a^3} \frac{erg}{cm^2} \quad (4)$$

and $-\frac{d}{da}E_{cas}(a) = F_{cas}$ of eq. (1).

The extension to spherical and other geometries involves difficult mathematics. Since the formal evaluation involves delicate cancelations, often even the sign of F_{cas} cannot be guessed prior to the complete calculation. The purpose of this letter is three fold:

- (1) We suggest a “polarized” version of the casimir set-up in which the force is controlled by relative rotation of the plates.
- (2) We discuss the magnetic part of the Casimir force and speculate on its modification for superconducting rings.
- (3) We present general suggestive arguments for an a priori prediction of the sign of the Casimir effect.

(1) The two independent polarizations $E_x \neq 0, E_y = 0; E_y \neq 0, E_x = 0$ contribute equally to F_{cas} . By an appropriate “twist” we can however use these polarizations to generate a variable, controlled, F_{cas} .

The Casimir force can be also derived by evaluating the pressure imbalance due to reflection of vacuum modes off the outside surfaces of the plate and of the (fewer) internal modes off the inside surfaces [3]. This derivation is completely rigorous. It amounts to exchanging the $-\frac{d}{da}$ and mode summation/integration. It is however very appealing and suggestive. Thus, F_{cas} is almost unchanged if a mesh of conducting wires of radius d with mesh size $b \leq a$ is substituted for the plates (see fig. (1)). The point is that the modes most relevant for the Casimir effect those with $\lambda \approx a$ will be equally well reflected, provided that the ratios $a/b \equiv r_1 > 1$ and $b/d \equiv r_2 > 1$ are not too large. [4]

If however we have only vertical (horizontal) conducting wires on both sides only the \hat{e}_y (\hat{e}_x) polarized modes will be reflected and we readily find:

$$F_{cas}(\text{parallel wires}) = \frac{1}{2}F_{cas} \quad (5)$$

Next consider wires on the left and right that are crossed at a right angle with vertical wires on the right. The \hat{e}_y modes impinging from the right on the outer part of the right handed mesh will be exactly balanced by \hat{e}_y modes impinging from the left which freely coast through the left mesh with horizontal wires. Thus

$$F_{cas}(\text{orthogonal wires}) = 0 \quad (6)$$

This indeed is expected directly in the original method of derivation: For crossed orthogonal wires, neither the \hat{e}_y nor the \hat{e}_x modes are confined to $0 \leq z \leq a$. Consequently there is no k_z quantization and no Casimir forces.

Consider next the case when the wires on the left make any angle θ relative to the (say vertical) wires on the right. The \hat{e}_y modes impinging from the left will now be partially reflected, with probability $\cos^2\theta$, from the left mesh. This then causes a pressure imbalance and a resulting inward directed pressure on the right mesh proportional to $\cos^2\theta$. Since Eq. (5) corresponds to the limiting case $\theta = 0$ we have:

$$F_{cas}(\text{wires at angle } \theta) = \frac{1}{2}\cos^2\theta F_{cas} \quad (7)$$

The last equation is the most important result of this letter. It suggests that a controlled F_{cas} signal could be expected as circular parallel rings with wires strung along them are rotated with respect to each other. [5]

(2) The Casimir force between two square conducting loops of wire thickness $\approx a$ of side a a distance a apart is, on dimensional grounds:

$$F_{cas}(loop-loop; a, a) \approx \frac{\xi \hbar c}{a^2} \quad (8)$$

This can be further motivated and $\xi \approx \frac{1}{40}$ can be roughly estimated by reconsidering the mesh of fig. (1) in the limit $b \rightarrow a$ as follows: since $\lambda \approx a \geq b$ is still marginally satisfied, there will be substantial reflection of the relevant $\lambda \approx a$ modes. If reflection is reduced by $\frac{1}{2}$ then the mesh-mesh force per unit area is $\frac{1}{2}F_{cas}$.

However for $a = b$, the mesh-mesh force comes mainly from the attraction of opposing single squares in the two meshes, fig. (2). Further more we can under these circumstances approximate the mesh-mesh force per cm^2 , by the sum of $\frac{1}{a^2}$ loop-loop interactions - leading to Eq. (8).

The derivation by mode counting and energy subtraction suggests that the Casimir effect is equally magnetic or electric since the electric energy and the magnetic energy make equal contributions to $\frac{\hbar\omega}{2}$. None-the-less, the formal boundary conditions are the end-products of complicated underlying processes involving charges and currents induced by the E and B fluctuations.

Consider then two parallel conducting rings of size a and at a distance a apart. The magnetic vacuum fluctuations include closed B lines which may link none of the rings, some that link one or the other and some which link both rings (see fig. (3)). The last set is relevant to forces between the two rings. It will induce, by Faraday's law, parallel currents in the two rings. Thus, regardless of the sign of the B fluctuation and of the ensuing circulating current, the resulting current current forces will be attractive. [6]

How will this force be modified if the rings become superconducting? In this case, B fields smaller than B_{crit} cannot penetrate the superconductor if these fields oscillate at a frequency ω lower than the critical frequency, i.e. if:

$$\omega \leq \omega_c = \frac{kT_c}{\hbar} \quad (9)$$

For high T_c superconductors both B_c and ω_c are larger, thereby excluding a wider range of vacuum fluctuations from entering the superconductor. In particular, if we put the rings at a distance a , such that $a \geq \frac{c}{\omega_c}$, the geometric mode cut-off in the Casimir effect, $\omega \leq \omega_{max} \approx \frac{c}{a}$, will automatically ensure eq. (7). Also, the magnitude of the relevant B fluctuations on this scale $B^2 \approx \frac{\hbar^2}{c^2 a^4}$ is small enough to ensure $B \leq B_{crit}$.

The superconducting rings will then impose a new important integral constraint, namely that the total fluxes

threading the various superconducting rings must be integer multiples of the flux quantum: $\Phi = n\Phi_0 = \frac{n\hbar c}{e}$. Since the fluctuation of interest are of scale $\lambda \approx a$ the above current - current forces on the various segments of the rings add coherently corresponding to the net current flow in the rings R_1, R_2 . If there is no net global flux change in the rings due to the vacuum fluctuation there will be - in this approximation - no net current and no net force.

The quantization condition implies however a strong exponential suppression for all $n \neq 0$ sectors. Thus if we have a fluctuation with roughly constant B on scale a :

$$\pi B a^2 \approx n\Phi_0 \approx \frac{n\hbar c}{e} \quad (10)$$

The action of such a configuration will therefore be:

$$A = \int (cB)^2 d^3x dt \approx \pi^2 c^2 B^2 a^3 \frac{a}{c} = \pi^2 c (Ba^2)^2 = \frac{cn^2 \hbar^2}{e^2} \quad (11)$$

The exponential suppression will then be

$$e^{-\frac{A}{\hbar}} \approx e^{-\frac{n^2}{\alpha_{em}}} \quad (12)$$

rendering such fluctuation and the attendant magnetic Casimir forces completely negligible.

There is an amusing similarity between this suppression and that of instanton tunneling or the probability of exciting classical configurations with $O(\frac{1}{\alpha})$ coherent photons around the $n = 0$ sector by vacuum fluctuation, i.e. in perturbation theory. A particularly interesting example is the creation of a monopole anti-monopole pair which generates precisely a flux $= \Phi_0$. [7]

The above considerations suggest that if the Casimir force between conducting rings is constantly monitored as the temperature of the system is lowered below the superconducting critical temperature, then the quenching of part of the Casimir force due to the magnetic-current inducing fluctuation reduces the observed force. Even for $T_c \approx 100^0 \text{ Kelvin} \approx 10^{-2} \text{ ev}$, the highest superconducting temperature to date the minimal distance $a \geq \frac{\hbar c}{\omega_c}$, $\omega_c \approx 10^{-2} \text{ eV}$ is 20μ . At such a distance the full ordinary Casimir force per cm is $\approx 10^{-7}$ dyne. This is a very small force, comparable to that exerted on the tip of a tunneling force microscope due to a single van-der Waals bond!

(3) The Casimir forces can be viewed as arising from \mathbf{E} vacuum fluctuations on a scale $\lambda \approx a$ inducing opposite sign patches of charge density on the opposite plates (see fig. 4). The attractive electrostatic force between such patches yields the negative E_{cas} of eq. (1).

The interpretation of the Casimir energy as the electro(magneto)static interaction of the induced charges and currents is inspired by the original paper of Casimir and Polder [8]. The latter introduced the "retarded" r^{-7} potentials between neutral atoms at large distances.

In the pre-Feynman-diagram nomenclature used there, the leading r^{-6} contribution of Coulomb-Coulomb second order perturbation cancels against part of the transverse photon-Coulomb, third order perturbation. The remaining part of the latter contributes the r^{-7} force. It exactly features an induced charge distribution, say induced dipole, interacting via Coulombic electrostatic interactions. This interpretation is however of heuristic nature. In particular, it cannot readily account for the repulsive induced electric magnetic-interaction [9].

Consider next a general “conductor” consisting of the union of n surfaces S_1, \dots, S_n . Let $\rho_{ind}(r, t)$ and $\mathbf{j}_{in}(r, t)$ with $r \in S_i$ be the induced charge and current distributions. The electrostatic Casimir energy is then: $\int d^3 r d^3 r' \frac{\rho_{ind}(r) \rho_{ind}(r')}{|r - r'|}$.

We can readily verify that $E_{cas}\{S_i\} \geq 0$ by transforming the electrostatic and magnetostatic energies into $\int d^3 r (\mathbf{E}_{ind}^2 + \mathbf{B}_{ind}^2) \geq 0$. Next let us consider a uniform dilation $r \rightarrow \lambda r$ sending our original set of conductors into dilated surfaces $S_i \rightarrow \lambda S_i$ with dilated relative distances. From dimensional arguments the casimir energy for the new surfaces is related to that of the old set by:

$$E\{\lambda S_i\} = \frac{1}{\lambda} E\{S_i\} \quad (13)$$

The generalized force conjugate to such a “displacement” $F_\lambda = -\frac{\partial}{\partial \lambda} E\{\lambda S_i\}$ is therefore always positive and the system as a whole tends to dilate. For simple geometries such as a sphere or cylinder this implies repulsive Casimir forces [11].

There is no conflict with the attraction of the two Casimir plates - the force in question is conjugate only to the relative separation and only the mutual interaction say $\int d^3 r d^3 r' \rho_L(r) \rho_R(r') / |r - r'|$ with ρ_L (ρ_R) the induced density on left (right) plate is therefore manifest. Had we included also the self interaction (corresponding to the $\rho_L \rho_L + \rho_R \rho_R$ term), positivity would be regained. However the forces associated with the latter are “mute” strains inside the plate.

We can also argue on general grounds that for two objects A, B , having similar shapes and composition at a distance R larger than their size a , the Casimir force is attractive. To this end we note that all Casimir-Older and Casimir forces are, in the final analysis, describable by two photon exchange diagrams. The analysis of Ref. [10] shows that $V(R) \propto -\int_0^\infty \frac{\sigma(t) e^{-\sqrt{t} R}}{t} dt$ with $\sigma(t)$ the t -channel discontinuity of the $A\bar{A} \rightarrow \gamma\gamma \rightarrow B\bar{B}$ amplitude. For $A = B$, the latter is inherently positive and the force is attractive even if A and B are macroscopic objects.

I am indebted to C. K. Au, A. Casher, J. Milana and J. Sucher for many discussions. I have particularly benefited from and enjoyed many most helpful discussions with Vel Hushwater. This work was supported in part by DOE Grant DOE-FG02-93ER-40762.

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FIG. 1. The two wire meshes replacing the two Casimir plates. The mesh–mesh distance a , the mesh square size b , and wire radius d satisfy $a > b > d$ but with not too large $a/b, b/d$ ratios.

FIG. 2. Two opposing squares of side $b = a$ from the two original meshes.

FIG. 4. An \mathbf{E} fluctuation inducing opposite charges on the two conducting Casimir plates.

FIG. 3. Two conducting rings R_1 and R_2 (the cross-hatched rings) and some \mathbf{B} flux lines. \mathbf{B}_{00} indicated by broken line interlocks none of the rings. \mathbf{B}_{10} indicated by one continuous line interlocks R_2 but not R_1 . \mathbf{B}_{11} is a double closed line interlocking both R_1 and R_2 .







